Zero-energy vortex bound states in noncentrosymmetric superconductors

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We consider bound states at the vortex core of a noncentrosymmetric superconductor. We show that despite the mixing of singlet and triplet order parameters zero-energy states survive within certain parameter space as in vortices of some chiral p-wave states.

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Noncentrosymmetric superconductors, such as $CePt_3Si$, Li_2Pt_3B, and $Mg_{10}Ir_{19}B_{16}$, have drawn a great deal of interest in the past few years.[1–3] Microscopically, the degeneracy among the two pseudospin states at a given momentum, connected by the composite operations of time-reversal and inversion transformations, is lifted since the latter symmetry is broken. Hence, Cooper pairs can no longer be classified in terms of singlets versus triplets nor s-wave versus p-wave.[4] In $CePt_3Si$, for example, the superconducting state is believed to have a triplet order parameter $\mathbf{d}(\mathbf{k})\propto(k_x\hat{y}-k_y\hat{x})$ in addition to a singlet one.[1] It has been proposed that this mixing is responsible for some of the seemingly contradictory behaviors of these superconductors.[2,5] Moreover, some intriguing broken symmetry properties of these superconductors have been predicted.[6]

Recently, bound state[7] within the odd winding number vortex of the p-wave k_x+ik_y superfluid has been a hot topic since among them there is a localized zero-energy state,[8–11] which, in terms of its associated creation operator, is a self-Hermitian Majorana fermion.[12] Proposals that utilize such zero-energy states for implementing the topological quantum computation[9,10] are based on its unique properties including robustness against perturbations from deformations of order parameters and nonmagnetic impurities.[11–15]

It is well known that the bound-state spectrum associated with an isolated vortex with winding 1 in an s-wave superconductor is $E=(n+\frac{1}{2})\omega$ with the energy scale ω of order Δ^2/E_F (Ref. 7) (here n is an integer, Δ the energy gap, and E_F the Fermi energy), while the spectrum in the A phase of p-wave superfluid ^3He is given by $E=n\omega$, which includes a state at zero energy.[8] In this Brief Report, we shall study the vortex bound states in noncentrosymmetric superconductors described by a mixture of singlet and triplet order parameters, which is intermediate between the previous two examples. First, we demonstrate that there are two such zero-energy states (for each p_z, to be defined later) corresponding to the pure p-wave order parameter with $\mathbf{d}(\mathbf{k})\propto(k_x\hat{y}-k_y\hat{x})$ presenting the combined up and down equal-spin pairings. Thus the two states differ by spin orientation, and at first sight, would be coupled by the additional s-wave order parameter and hence acquire finite energies. However, we shall show that as long as the s-wave component is smaller than a critical value the zero-energy states survive. Besides, we note that this condition is identical to the existence of a nodal gap. In addition, we consider a Rashba spin-orbital interaction. We shall show that the zero-energy states again survive.

The creation operator $\alpha^\dagger$ for a quasiparticle excitations in an inhomogeneous superconductor is a linear combination of electronic annihilation and creation field operators $\psi_{\uparrow,\downarrow}(\vec{r})$ and $\psi^\dagger_{\uparrow,\downarrow}(\vec{r})$,

$$\alpha^\dagger=\int_{\vec{r}}[u_\uparrow(\vec{r}),u_\downarrow(\vec{r})]\begin{bmatrix}\psi^\dagger_\uparrow(\vec{r})\\ \psi^\dagger_\downarrow(\vec{r})\end{bmatrix}+[v_\uparrow(\vec{r}),v_\downarrow(\vec{r})]\begin{bmatrix}\psi_\uparrow(\vec{r})\\ \psi_\downarrow(\vec{r})\end{bmatrix}, \tag{1}$$

satisfying $[H_{\rm eff},\alpha^\dagger]=\epsilon\alpha^\dagger$, where $H_{\rm eff}$ is the effective mean-field Hamiltonian and ϵ is the quasiparticle energy. The coefficients $(\hat{u},\hat{v})=(u_\uparrow,u_\downarrow,v_\uparrow,v_\downarrow)$ satisfy the Bogoliubov-deGennes (BdG) equation,

$$\begin{pmatrix}H_0 & \Pi\\ \Pi^\dagger & -H_0^*\end{pmatrix}\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}=\epsilon\begin{pmatrix}\hat{u}\\ \hat{v}\end{pmatrix}, \tag{2}$$

where, for the ordinary cases, $H_0=-\frac{\nabla^2}{2m}-E_F$ is the kinetic energy (we shall add the possible Rashba interaction later). Π is a two-by-two matrix due to the pairing. In the singlet case Π represents $\Delta_s(\vec{r})(i\sigma_2)$. In the triplet case $\Pi=\frac{1}{2}(\vec{\nabla}\cdot\vec{D})+\vec{D}\cdot\vec{\nabla}$. The vector $\vec{D}=-i\vec{\nabla}_{\mathbf{k}}\Delta(\vec{r},\mathbf{k})$, where $\Delta(\vec{r},\mathbf{k})$ is the order parameter. By the conventional notation, $\Delta(\vec{r},\mathbf{k})=\Delta_p(\vec{r})\mathbf{d}(\mathbf{k})\cdot\vec{\sigma}(i\sigma_2)$. Π is then a sum of the above two when both order parameters are present.

We first consider the simpler case (cf., e.g., Ref. 10) where the order parameter is of pure p-wave character with $\mathbf{d}(\mathbf{k})=(k_x\hat{y}-k_y\hat{x})/p_F$. [Here $p_F\equiv(2mE_F)^{1/2}$]. We shall show that for each value of momentum along the vortex line p_z less than p_F there are two zero-energy states with the associated wave function $[\hat{u}(\vec{r}),\hat{v}(\vec{r})]^T$ given by

$$(1,0,-1,0)^T R_1(\rho)e^{ip_z z},$$
$$(0,e^{i\phi},0,-e^{-i\phi})^T R_2(\rho)e^{ip_z z}, \tag{3}$$

in cylindrical coordinates $\vec{r}=(\rho,\phi,z)$, where the radial functions $R_{1,2}$ are independent, finite, and decaying at infinity.

For an isolated vortex line with winding 1, the order parameter can be expressed as $\Delta_p(\vec{r})=\Delta_p(\rho)e^{i\phi}$ where we shall choose the gauge where $\Delta_p(\rho)$ is real and positive. The coupling Π in Eq. (2) is then

$$i\frac{\Delta_p(\rho)}{p_F}\begin{pmatrix}e^{-i\phi/2}\left(\partial_\rho-\frac{i}{\rho}\partial_\phi\right)e^{i\phi/2} & 0\\ 0 & e^{i3\phi/2}\left(\partial_\rho+\frac{i}{\rho}\partial_\phi\right)e^{i\phi/2}\end{pmatrix}. \tag{4}$$

$\Delta_p(\rho)$ is zero at $\rho=0$ and increases toward its asymptotic value Δ_0 within a range of coherence length $\xi\gg p_F^{-1}$. In principle we should add also, respectively, $\frac{i}{2p_F}\partial_\rho\Delta(\rho)$ and $\frac{i}{2p_F}e^{2i\phi}\partial_\rho\Delta(\rho)$ to the upper left and lower right elements in Eq. (4) due to the $\vec{\nabla}\cdot\vec{D}$ term in Π, but it can be shown that since these terms are regular $\rho\to 0$ and vanish necessarily as $\rho\to\infty$ they do not affect the arguments below, and so we would not show them explicitly to simplify the equations. It is clear from Eq. (4) that the BdG, Eq. (2), for excitations with up and down spins become decoupled. For the states with $\epsilon=0$, we denote the two independent excitations by $\alpha_1^\dagger$ and $\alpha_2^\dagger$. For $\alpha_1^\dagger$ with up spin, the wave functions can be factored into $u_\uparrow(\vec{r})=e^{ip_z z}u_\uparrow(\rho)$ and $v_\uparrow(\vec{r})=e^{ip_z z}v_\uparrow(\rho)$. The equations for $u_\uparrow+v_\uparrow$ and $u_\uparrow-v_\uparrow$ become decoupled;

$$\left[\frac{1}{2m}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}\right)+\tilde{\mu}\mp\frac{\Delta_p(\rho)}{p_F}\left(\frac{d}{d\rho}+\frac{1}{2\rho}\right)\right](u_\uparrow\pm v_\uparrow)=0, \tag{5}$$

where $\tilde{\mu}(p_z)\equiv E_F-\frac{p_z^2}{2m}$ is an effective chemical potential. Here we assume $p_z<p_F$, and thus $\tilde{\mu}>0$. Two properties about Eq. (5) have to be noted here. First, for each of the above equation the presence of a regular singular point at the origin forces the general solutions to be divergent at the origin. Second, the equation corresponding to $u_\uparrow+v_\uparrow$ with a minus sign in front of $d/d\rho$ has its both solutions unbounded at infinity, whereas the equation for $u_\uparrow-v_\uparrow$ has two solutions decaying as $\rho\to\infty$. The above can be shown by substituting $e^{ip_\parallel\rho}$ into the asymptotic form of Eq. (5) where terms proportional to $1/\rho$ can be neglected and $\Delta_p(\rho)$ is replaced by Δ_0. Then we have $-x^2+\frac{\mu}{E_F}\mp i\frac{\Delta_0}{E_F}x=0$, where $x\equiv p_\parallel/p_F$. For the lower sign appropriate to $u_\uparrow-v_\uparrow$, the solutions for x and hence $p_\parallel$ is given by $p_F[+i\frac{\Delta_0}{2E_F}\pm\sqrt{\frac{\tilde{\mu}}{E_F}}]$ (assuming a weak-coupling superconductor and thus $\Delta_0\ll E_F$) with positive imaginary parts (if $\tilde{\mu}>0$). Hence $u_\uparrow-v_\uparrow$ has a pair of decaying solutions $\chi_1(\rho)$ and $\chi_2(\rho)$. For the upper sign appropriate to $u_\uparrow+v_\uparrow$, we only have two growing solutions that must be rejected and so $u_\uparrow(\rho)+v_\uparrow(\rho)=0$ must hold. Now both solutions $\chi_1(\rho)$ and $\chi_2(\rho)$ in general contain a divergence of $\ln\rho$ as $\rho\to 0$. Nevertheless, a finite solution as $\rho\to 0$ can be found by an appropriate linear combination so that we can write the zero-energy state as the first line in Eq. (3), $R_1(\rho)=a\chi_1(\rho)+b\chi_2(\rho)$ where the coefficients are determined to cancel $\ln\rho$ near the origin. (For $|p_z|>p_F$, $\tilde{\mu}<0$, then both $u_\uparrow\pm v_\uparrow$ have a single solution, which decays at infinity, and thus no zero-energy state is allowed; cf., e.g., Refs. 10 and 12).

The state associated with $\alpha_2^\dagger$ can be obtained in a similar manner. The factorization is $[u_\downarrow(\vec{r}),u_\downarrow(\vec{r})]=e^{ip_z z}[e^{i\phi}\tilde{u}_\downarrow(\rho),e^{-i\phi}\tilde{v}_\downarrow(\rho)]$. The equations are also decoupled for $\tilde{u}_\downarrow\pm\tilde{v}_\downarrow$ as in previous case, but the kinetic energy contains an additional term $-1/\rho^2$, which causes a divergence of $1/\rho$ for the solution near the origin. Analogous procedure shows that there are two decaying solutions, $\eta_1(\rho)$ and $\eta_2(\rho)$, for $\tilde{u}_\downarrow-\tilde{v}_\downarrow$, but there are two exponentially increasing solutions for $\tilde{u}_\downarrow+\tilde{v}_\downarrow$. Therefore we have the solution as $[\hat{u}_2(\vec{r}),\hat{v}_2(\vec{r})]=e^{ip_z z}(0,e^{i\phi},0,-e^{-i\phi})R_2(\rho)$, where $R_2=c\eta_1+d\eta_2$ is a suitable linear combination to cancel the divergence of $1/\rho$. Now we obtain the eigenfunctions associated with the two zero-energy excitations. We note the relation,

$$\hat{u}_2=e^{i\phi}\sigma_1\hat{u}_1\kappa(\rho),\quad \hat{v}_2=e^{-i\phi}\sigma_1\hat{v}_1\kappa(\rho), \tag{6}$$

where $\kappa(\rho)\equiv R_2(\rho)/R_1(\rho)$, which will be useful later.

With the solutions, Eq. (3), obtained for a pure p-wave order parameter, Eq. (4), we are going to consider the effects of lacking inversion symmetry on Eq. (3). For simplicity of presentation, we shall consider separately the Rashba interaction and an admixture of singlet order parameter, but we only state the general conclusion at the end. First, we shall show via perturbation theory that a small Rashba interaction or a small s-wave order parameter would not destroy the zero-energy states. Then we shall consider the general magnitude of these two interactions.

Now we add a Rashba spin-orbital interaction $h=(-\alpha\hat{z}\times\mathbf{p}\cdot\vec{\sigma})$ to the kinetic-energy parts H_0 in Eq. (2). In cylindrical coordinates, h and its counterpart $-h^*$ associated with the hole sector can be written as

$$\pm\alpha\begin{pmatrix}0 & e^{\mp i\phi}\left(\partial_\rho\mp\frac{i}{\rho}\partial_\phi\right)\\ -e^{\pm i\phi}\left(\partial_\rho\pm\frac{i}{\rho}\partial_\phi\right) & 0\end{pmatrix}, \tag{7}$$

where the upper (lower) sign is for the electron (hole) sectors, respectively. The expectation values of the this spin-orbital interaction for either of the two states given above are obviously zero. The matrix element between the two states in Eq. (3) is proportional to the spatial integral of $\hat{u}_2^\dagger h\hat{u}_1-\hat{v}_2^\dagger h^*\hat{v}_1$, which, with Eq. (6), equals $\{\hat{u}_1^\dagger\sigma_1[e^{-i\phi}h+e^{i\phi}(-h^*)]\hat{u}_1\}$ times some function of ρ. Hence the matrix element is zero by explicit use of Eq. (7). Therefore, within perturbation, the two states in Eq. (3) are unaffected by the coupling from the spin-orbital interaction.

Now consider an additional a singlet pairing order parameter. We therefore add a term,

$$\Delta_s(\vec{r})=\begin{pmatrix}0 & e^{i\phi}\\ -e^{i\phi} & 0\end{pmatrix}\Delta_s(\rho). \tag{8}$$

to the pure triplet one in Eq. (4) and in Eq. (2). For mixing, which do not break time-reversal symmetry far from the vortex, the ratio $\lim_{\rho\to\infty}[\Delta_s(\rho)/\Delta_p(\rho)]\equiv\beta$ must be real. The expectation value of Eq. (8) in any given state in Eq. (3) is again obviously zero, and the matrix element of Eq. (8) between the two states in Eq. (3) involves the spatial integral of $\hat{u}_2^\dagger e^{i\phi}(i\sigma_2)\hat{v}_1+\hat{v}_2 e^{-i\phi}(-i\sigma_2)\hat{u}_1$, which, by using Eq. (6), is zero. Hence in the small β regime, the two zero-energy states remain.

The previous two paragraphs demonstrate that the two states in Eq. (3) remain according to perturbation theory. Now we consider the more general case. We shall show that there are two zero-energy states with general form

$$e^{ip_z z}[u_\uparrow(\rho),\ e^{i\phi}\tilde{u}_\downarrow(\rho),\ v_\uparrow(\rho),\ e^{-i\phi}\tilde{v}_\downarrow(\rho)]^T, \tag{9}$$

with $u_\uparrow=-v_\uparrow$ and $\tilde{u}_\downarrow=-\tilde{v}_\downarrow$ all finite and decaying at infinity, which survive the additional interactions.

First consider order-parameter mixing. The operator Π in BdG equation is a sum of Eqs. (4) and (8). The corresponding set of differential equations are

$$L_0 u_\uparrow-Pv_\uparrow-\Delta_s(\rho)\tilde{v}_\downarrow=0,$$

$$L_0 v_\uparrow - P u_\uparrow - \Delta_s(\rho)\tilde{u}_\downarrow = 0,$$

$$L_1 \tilde{u}_\downarrow - P \tilde{v}_\downarrow + \Delta_s(\rho) v_\uparrow = 0,$$

$$L_1 \tilde{v}_\downarrow - P \tilde{u}_\downarrow + \Delta_s(\rho) u_\uparrow = 0, \tag{10}$$

where the differential operators $L_n = \frac{1}{2m}(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{n^2}{\rho^2}) + \tilde{\mu}$ and $P = \frac{\Delta(\rho)}{p_F}(\frac{d}{d\rho} + \frac{1}{2\rho})$. Denoting $w_\uparrow^\pm = u_\uparrow \pm v_\uparrow$ and $w_\downarrow^\pm = \tilde{u}_\downarrow \pm \tilde{v}_\downarrow$, the above equations for the w^+'s are decoupled with the w^-'s. Writing $z_\pm^+ = w_\uparrow^+ \pm i w_\downarrow^+$ and $z_\pm^- = w_\uparrow^- \pm i w_\downarrow^-$, we arrive at

$$\left(L_0 + P - \frac{1}{4m\rho^2} \mp i\Delta_s(\rho)\right) z_\pm^- + \frac{1}{4m\rho^2} z_\mp^- = 0, \tag{11}$$

$$\left(L_0 - P - \frac{1}{4m\rho^2} \pm i\Delta_s(\rho)\right) z_\pm^+ + \frac{1}{4m\rho^2} z_\mp^+ = 0. \tag{12}$$

Although in general z_+^- (z_+^+) couples with z_-^- (z_-^+), we note that at infinity the above set of equations for each of $z_\pm^+$ and $z_\pm^-$ couples to itself only. Then for a given $p_z = p_F \cos\theta$ associated with the excitation, we can write $\tilde{\mu} = E_F \sin^2\theta > 0$. $z_\pm^-$ satisfies asymptotically for large ρ,

$$\left[\frac{1}{2m}\frac{d^2}{d\rho^2} + \frac{\Delta_0}{p_F}\frac{d}{d\rho} + E_F \sin^2\theta \mp i\beta\Delta_0\right] z_\pm^- = 0, \tag{13}$$

where one recalls that $\Delta_s(\rho \to \infty) = \beta\Delta_0$. Note that a similar equation as Eq. (13) except the positive coefficient in front of $d/d\rho$ is for the z^+'s. Again, take $e^{ip_\parallel \rho}$ as asymptotic solutions. In the weak-coupling limit, $p_\parallel$ for z^- are given by $p_F[i\frac{\Delta_0}{2E_F} \pm \sqrt{\sin^2\theta \pm i\beta\frac{\Delta_0}{E_F}}]$. The imaginary parts are therefore, for small Δ_0/E_F, $\frac{p_F}{2}\frac{\Delta_0}{E_F}(1 \pm \frac{\beta}{\sin\theta})$. For a given p_z and θ, there are four roots associated with $z_\pm^-$ with positive imaginary parts when $\frac{|\beta|}{\sin\theta} < 1$, which leads to four independent decaying solutions for $z_\pm^-$. The same arguments show that the solutions for $z_\pm^+$ are exponentially increasing as $\rho \to \infty$, and hence we must choose $w_\uparrow^+ = w_\downarrow^+ = 0$, that is, $u_\uparrow = -v_\uparrow$ and $\tilde{u}_\downarrow = -\tilde{v}_\downarrow$ as in Eq. (9). Therefore a general solution for the zero-energy state for Eq. (10) can be represented by the linear combination of the four decaying solutions,

$$\begin{pmatrix} u_\uparrow(\rho) \\ \tilde{u}_\downarrow(\rho) \\ v_\uparrow(\rho) \\ \tilde{v}_\downarrow(\rho) \end{pmatrix} = \sum_{i=1}^{4} c_i \begin{pmatrix} 1 \\ y_i(\rho) \\ -1 \\ -y_i(\rho) \end{pmatrix} f_i(\rho), \tag{14}$$

where the f's decays toward zero at infinity and so are the $y_i f_i$. From Eq. (10), the divergences near the origin are of the form $\ln\rho$ for the first row and $1/\rho$ for the second row. Now we have two equations determining the c's by which the respective divergence can be removed. In general, we can have two independent sets of $\{c_i\}$ satisfying the above, which in turn leads to two independent zero-energy states within the vortex core. A crucial consequence is drawn from the above arguments. For the relative pairing strength $|\beta| > 1$, $|\beta/\sin\theta| > 1$ so that the zero-energy states no longer survive, which we conclude that in addition to $\tilde{\mu} = 0$ $|\beta_c| = 1$ is another critical parameter. On the other hand, the zero-energy states exist at the core when $|\beta| < 1$. The density of such excitations (number per unit length of the vortex line) can be determined from the condition $|\beta/\sin\theta| < 1$. For a spherical Fermi surface, we obtain $\frac{2p_F}{\pi}\sqrt{1-\beta^2}$. We note that the energy gap for the bulk excitations are given by $\Delta_0|\sin\theta \pm \beta|$, hence the above critical value of θ for the existence of the $E=0$ vortex bound state corresponds to exactly the existence of a nodal line in one of the branches. This is reasonable as this is the value of p_z where two of the decaying solutions for $\sin\theta > |\beta|$ become extended, destroying the possibility of obtaining the solution Eq. (9), which converges both at $\rho \to \infty$ and 0.

Next we move on to the case when the spin-orbital interaction Eq. (7) is included in Eq. (2). With again the wave function in the form of Eq. (9), the zero-energy BdG equation can be written as

$$L_0 u_\uparrow - P v_\uparrow - \alpha\left(\frac{d}{d\rho} + \frac{1}{\rho}\right)\tilde{u}_\downarrow = 0,$$

$$L_0 v_\uparrow - P u_\uparrow - \alpha\left(\frac{d}{d\rho} + \frac{1}{\rho}\right)\tilde{v}_\downarrow = 0,$$

$$L_1 \tilde{u}_\downarrow - P\tilde{v}_\downarrow + \alpha\frac{d}{d\rho}u_\uparrow = 0,$$

$$L_1 \tilde{v}_\downarrow - P\tilde{u}_\downarrow + \alpha\frac{d}{d\rho}v_\uparrow = 0. \tag{15}$$

We can analyze these equations in the same manner as the previous case. At infinity the BdG equations become decoupled as

$$\left[\frac{1}{2m}\frac{d^2}{d\rho^2} + \left(\frac{\Delta_0}{p_F} \pm i\alpha\right)\frac{d}{d\rho} + E_F \sin^2\theta\right] z_\pm^- = 0, \tag{16}$$

where $\cos\theta = p_z/p_F$, and note there is a similar equation for $z_\pm^+$ except the overall positive coefficient associated with $d/d\rho$. $p_\parallel$'s associated with the asymptotic solution $e^{ip_\parallel \rho}$ have imaginary part $\frac{\Delta_0}{2E_F}(1 \pm \frac{\alpha/v_F}{\sqrt{(\alpha/v_F)^2 + \sin^2\theta}})$, which is positive for all θ. (Here $v_F \equiv p_F/m$.) On the contrary, the corresponding $p_\parallel$ for $z_\pm^+$ have only negative imaginary parts. Applying the same arguments as before, the zero-energy bound states survive under any magnitude of the spin-orbital interaction. One can understand this result by the fact that the size of the Fermi surfaces at p_z for the two branches are given by $p_{F\pm} \equiv [(2m\tilde{\mu}) + (m^2\alpha^2)]^{1/2} \pm m\alpha = p_F[\sqrt{(\frac{\alpha}{v_F})^2 + \sin^2\theta} \pm \frac{\alpha}{v_F}]$, which remain finite for arbitrary large value of α.

The above analysis can be generalized to the case where both α and β are finite. We find that a pair of $E=0$ states exist if one has both $\sqrt{(\frac{\alpha}{v_F})^2 + \sin^2\theta} + \frac{\alpha}{v_F} + \beta > 0$ and $\sqrt{(\frac{\alpha}{v_F})^2 + \sin^2\theta} - \frac{\alpha}{v_F} - \beta > 0$. We note that in the helicity basis [spin quantization axis along $(\hat{z} \times \mathbf{p})$], the order parameter on the $\pm$ branches of the Fermi surfaces are given, respectively, by $\Delta_p \frac{p_{F\pm}}{p_F} \pm \Delta_s = \Delta_p[\sqrt{(\frac{\alpha}{v_F})^2 + \sin^2\theta} \pm \frac{\alpha}{v_F} \pm \beta]$, hence the existence or absence of the $E=0$ bound states is determined by the relative sign of the order parameter on these two Fermi surfaces.

The solutions in Eq. (9) taking the lack of inversion into account are as robust as that of Eq. (3) in p-wave superfluids. It is evident that the local charge density is zero since the solutions have equal magnitudes for electron and hole excitations of the same spin projection. Hence the states are not susceptible to nonmagnetic impurities. Similarly, the states are not affected by the Zeeman magnetic field along $\hat{z}$ again due to the particle-hole symmetry. Furthermore, they are also not altered by exchange or Zeeman fields in the in-plane directions; for the azimuthal dependence in each of Eq. (9) leads to zero matrix elements of the local spin-density operator among the states.[16]

With the two general independent solutions of the form in Eq. (9), the corresponding creation operators are not necessarily self-Hermitian. Here we show that a set of two independent Majorana fermions can be built from them. We first demonstrate this for $p_z=0$. From the two linearly independent solutions, one first construct two orthonormal wave vectors $(\hat{u},\hat{v})$ and thus two corresponding operators $\alpha_1^\dagger$ and $\alpha_2^\dagger$. From orthonormal properties one then have $\{\alpha_1^\dagger,\alpha_1\}=\{\alpha_2^\dagger,\alpha_2\}=1$ and $\{\alpha_1^\dagger,\alpha_2\}=0$. Since $[H_{\mathrm{eff}},\alpha_i^\dagger]=E_i\alpha_i^\dagger=0$, we also have $[H_{\mathrm{eff}},\alpha_i]=-E_i\alpha_i=0$ for $i=1,2$. Since we have only two linearly independent solutions to Eq. (2) of zero energies, $\alpha_{1,2}^\dagger$ must just be a linear combinations of $\alpha_{1,2}$. We denote this in matrix notation as $\alpha^\dagger=\mathbf{C}\alpha$, where $\mathbf{C}$ is a 2×2 matrix. We shall find the transformation $\gamma^\dagger=\mathbf{W}\alpha^\dagger$ such that the γ's are independent Majorana fermion operators, that is, $\gamma_i^\dagger=\gamma_i$ for $i=1,2$ and $\{\gamma_1,\gamma_2^\dagger\}=0$. By choosing $\mathbf{W}$ as unitary, we can assure the conditions of normalization and orthogonality for the γ's. It remains only to make the γ's self-conjugate. From $\{\alpha_i,\alpha_j^\dagger\}=\delta_{ij}$, where $i,j=1$ or 2, one can show that $\mathbf{C}$ is unitary. Furthermore, $\alpha^\dagger=\mathbf{C}\mathbf{C}^*\alpha^\dagger$ and thus $\mathbf{C}^{-1}=\mathbf{C}^*$. Since $\mathbf{C}$ is unitary, $\mathbf{C}$ is also symmetric. Hence we can write $\mathbf{C}=e^{i\lambda}e^{i\omega\hat{n}\cdot\vec{\sigma}}$, where λ is real and $\hat{n}$ is perpendicular to $\hat{y}$. We thus have $\gamma^\dagger=\mathbf{W}\mathbf{C}\alpha=\mathbf{W}\mathbf{C}\mathbf{W}^T\gamma$, which would equal γ if we choose $\mathbf{W}=e^{-i\frac{\omega}{2}\hat{n}\cdot\vec{\sigma}}e^{-i\lambda/2}$. Thus the operators $\gamma_1^\dagger,\gamma_2^\dagger$ constitutes a set of two independent Majorana fermions. The constructions of Majorana fermions for finite p_z's can proceed in a similar manner if we replace the $e^{ip_z z}$ factors in the wave functions by $\cos(p_z z)$ and $\sin(p_z z)$.

As demonstrated already in, e.g., Refs. 9 and 12 the dimension of Hilbert space of a Majorana fermion is $\sqrt{2}$, that is, each two Majorana fermions combines to form one fermionic state with two degrees of freedoms (occupied or empty). Hence, for our system with n_v vortices per unit area, we have a residual entropy density $n_v\frac{p_F}{\pi}\sqrt{1-\beta^2-\frac{2\alpha}{v_F}\beta}\ln 2$. This ground-state degeneracy is lifted only by the finite overlap between the vortices.[14] The resulting energies are thus exponentially small in the vortex spacings. The existence of this residual entropy can be used to demonstrate the existence of $E=0$ vortex bound states, as well as be a measure of the mixing of the two superconducting order parameters.

In conclusion, we have considered the vortex bound states in a noncentrosymmetric superconductors, in particular for an order parameter appropriate to $CePt_3Si$. We demonstrated that the zero-energy states exist only for certain range of p_z values depending on the magnitudes of the singlet versus the triplet order parameters.

Note added. Recently, we became aware of Ref. 17, which however discusses a very different aspect of vortex bound states of noncentrosymmetric superconductors.

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